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CATASTROPHIC RISK MANAGEMENT: STOCHASTIC HYBRID MODEL TO CALCULATE THE LOSS INDEX TRIGGER FOR CATASTROPHE BONDS (CAT BONDS). ADJUSTMENT USING EVOLUTIONARY STRATEGIES

LA GESTIÓN DEL RIESGO CATASTRÓFICO: MODELO HÍBRIDO ESTOCÁSTICO PARA CALCULAR EL ÍNDICE DE PÉRDIDAS DESENCADENANTE DE LOS CAT BONDS. AJUSTE MEDIANTE ESTRATEGIAS EVOLUTIVAS

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ABSTRACT

Purpose: This paper develops a stochastic model to calculate the loss index trigger for catastrophe bonds as alternative instruments for the management of major insured risks, such as natural catastrophe.

Methodology: The underlying loss index of catastrophe bonds is the aggregate catastrophe losses reported before the end of certain period. The catastrophe severity is defined as the sum of two random variable: the reported loss amount and incurred-but-not-yet-reported loss amount, and the central hypothesis is that the latter decreases proportionally to a linearly increasing function up to a certain time and constant thereafter, called the hybrid claim reporting rate. Randomness in the reporting process is represented by a geometric Brownian motion in the claim reporting rate. The validity of the proposed model is evaluated by estimating its parameters using machine learning techniques (specifically, evolutionary strategies, ES).

Findings: The results shows that the model accurately captures the uneven behavior of the claim reporting process over time and therefore correctly describes the catastrophic claims reporting process.

Originality: The model proposed allows for an easy calculation of catastrophic loss indexes, thus facilitating the pricing of loss index-triggered Cat bonds. This translates into better catastrophe risk management for both insurance and reinsurance companies, as well as for

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those companies that diversify their portfolios with this type of financial instruments. The simplicity of the presented model facilitates parameter estimation and simulation.

Keywords: Catastrophic risk management, Catastrophe bonds, Reported loss amount, Incurred-but-not-yet-reported loss amount, Hybrid claim reporting rate, Evolutionary strategies

RESUMEN

Objetivo: Este artículo desarrolla un modelo estocástico para calcular el índice de pérdidas desencadenante de los bonos catastróficos como instrumentos alternativos de gestión de grandes riesgos asegurados, como las catástrofes naturales.

Metodología: El índice de pérdidas subyacente de los bonos catastróficos es el total de pérdidas por catástrofes declaradas antes del final de un periodo determinado. La cuantía total de la catástrofe se define como la suma de dos variables aleatorias: cuantía declarada de siniestros y cuantía de siniestros pendiente de declarar y se supone que esta variable disminuye proporcionalmente a una función linealmente creciente hasta un determinado momento y constante a partir de entonces, denominada tasa híbrida de declaración de siniestros. La aleatoriedad en el proceso de declaración se representa mediante un movimiento browniano geométrico en la tasa de declaración de siniestros. La validez del modelo propuesto se evalúa estimando sus parámetros mediante técnicas de aprendizaje automático (en concreto, estrategias evolutivas, ES).

Resultados: Los resultados muestran que el modelo captura con precisión el comportamiento desigual del proceso de declaración de siniestros a lo largo del tiempo describiendo correctamente el proceso de declaración de siniestros catastróficos.

Originalidad: El modelo permite calcular fácilmente los índices de siniestralidad catastrófica, facilitando así la tarificación de los Cat bonds. Esto se traduce en una mejor gestión del riesgo catastrófico tanto para aseguradoras y reaseguradoras, como para aquellas empresas que diversifican sus carteras con este tipo de instrumentos financieros. La simplicidad del modelo facilita la estimación de parámetros y la simulación.

Palabras clave: Gestión del riesgo catastrófico, Bonos sobre catástrofes, Cuantía declarada de siniestros, Cuantía de siniestros pendiente de declarar, Tasa de declaración de siniestros mixta, Estrategias evolutivas

1. INTRODUCTION

Over the last decades, there has been a growing tendency in the repercussion of natural catastrophes such as hurricanes, earthquakes, floods, among others. This is due to many factors and, despite some people don’t agree, most of them caused by human race. That climatic change is increasing the frequency of catastrophes seems evident, but it’s not the only problem we have to confront. Human society is growing uncontrollable and disorganized. While the first world has some comfort, that only signify a short percentage of the total population, the rest of people live following the rules the first world imposes. They produce what we want, buy what we throw and grow as we let them. These are the main reasons why they live in densely build up areas. All these facts suppose that when a catastrophe occurs, the consequences for the world are each time more sever as much in terms of human lives and in terms of economic losses. In 2022 alone, global natural catastrophe losses were $132 billion in insured damages (AON, 2023) and caused more than 12,000 deaths (Our world in data, 2024).

Before 1992, insurance companies were limited to paying the amount of damage caused by catastrophes (Polacek, 2018). But that year, a series of major catastrophes occurred in a short period of time, which collapsed the system, making it impossible for the conventional catastrophe insurance system to cover catastrophic events.

It was then that the Chicago Board of Trade launched CAT futures and CAT options to hedge against catastrophes (Board of Trade of the City of Chicago, 1992). These derivative financial instruments have as their underlying a catastrophe loss index and allow insurance companies to assume the risk of catastrophic events and reinsure the damages caused by catastrophes. These insurance linked derivatives have been evolving and currently catastrophe bond issues are the form of securitization that has been most developed and used by the insurance market in recent years.

Cat bonds are debt instruments that provide the insurance industry with access to a new source of risk hedging through the capital markets. (Pérez-Fructuoso, 2005). They are highly profitable and although their structure is like that of traditional bonds, their performance is conditional on the occurrence of a certain triggering event, the parameters of which are fixed in the issue. These bonds are sponsored by insurance companies, reinsurers, governments, or other institutions that cede part or all their catastrophe risk to a Special Purpose Vehicle (SPV). In return, the SPV writes a traditional reinsurance policy with the sponsor and seeks financing (by issuing bonds) in the capital market, which in turn acts as a counterparty to the reinsurance agreement.

The funds obtained from the bond issue and the reinsurance premium are invested by the SPV in short-term, high-return assets. These assets are deposited in a collateral account, which guarantees the transaction and generates sufficient resources to meet the risks covered by the reinsurance and to pay coupons to investors. The profits generated in this account are exchanged for LIBOR, with a swap counterparty that is highly valued by rating agencies. Through this swap, bonds are converted into floating rate securities so that interest rate risk is largely eliminated. During the bond's life, the periodic interest paid by the SPV to investors is obtained from the combination of two components: the premiums paid by the sponsor for reinsurance coverage and the LIBOR yield generated by the bond's principal, which is guaranteed by the swap counterparty. Then, at maturity, if the catastrophic event covered by the contract does not occur, the principal is returned to investors as with other fixed income investments. However, if the bond triggering event occurs, investors will lose the interest and principal of the investment or part of it depending on the structure of the bond and the terms of the reinsurance contract.

2. OBJECTIVES

The most complicated aspect of creating a catastrophe bond is defining what triggers the capital loss. There are basically four types of triggers: indemnity, industry loss ratios, parametric and modeled loss ratios. And of these, industry loss ratios are the second most important, accounting for 19,9% of total issuance until November 2024 (22.4% of total issuance during 2023) (Artemis, 2024).

The modeling of these loss ratios to price catastrophe bonds has been discussed in a variety of scientific papers. The scientific literature reviewed uses geometric Wiener processes to model either the reported loss amount or the catastrophe loss index. This means that the intensity of reported loss grows exponentially over time. If Wiener processes are also combined with Poisson processes to represent the occurrence of new catastrophes, this intensity of reported loss becomes discontinuous because of the jumps generated by the Poisson process. However, empirical evidence indicates that at the beginning, immediately after the occurrence of the catastrophe, many claims are reported, i.e., the intensity of reporting is high, and as time goes by the intensity of claims associated with the catastrophe decreases until it is cancelled when there are no more claims to report. Therefore, the objective of this work is to adequately represent this reported loss amount to obtain more accurate loss index values and consequently lower losses for the issuers of these instruments and more realistic prices for the investors.

2.1. Literature Review: Related Work and Research Framework

Several authors have investigated the insurance-linked derivatives valuation. The method usually employed is the development of pricing models based on the hypothesis of geometric Brownian motion, to systematize the instantaneous reporting claims evolution, and to incorporate the possibility of major catastrophes occurring through Poisson processes.

Cummins and Geman (1995) pricing of the first generation of catastrophe pricing of the first generation of catastrophe, Cat futures and Cat options, traded at the Chicago Board of Trade. They defined the loss index as the sum of the claims associated with each catastrophe and used a geometric Brownian motion to represents the randomness of the claims reporting process, and a Poisson process that incorporates the jumps in the claim process due to the occurrence of new catastrophes. Geman and Yor (1997), follow a similar approach but use the diffusion process with jumps to directly model the loss rate of Property Claim Services (PCS) options. Aase (1999) develop a valuation model of catastrophe futures when the loss index follows a stochastic process containing jumps of random claim sizes at random time points of accident occurrence. This model is a particular case of the model created by Embrechts and Meister (1997) which represents the behavior of the catastrophic loss index through a mixture of compound Poisson processes and a random loss frequency. Baryshnikov, Mayo and Taylor (2001), using continuous trading and risk neutrality, price the catastrophe bond using a double compound Poisson process to capture the different characteristics of catastrophe dynamics. Burnecki and Kukla (2003) apply the results of Baryshnikov, Mayo and Taylor (2001) to calculate non-arbitrage prices of a zero-coupon and coupon CAT bond. Muermann (2003) introduces the concept of actuarial consistency and derives a representation of the prices of non-arbitrage catastrophe derivatives (Cat futures and Cat options) written on an underlying loss index that is modeled as a compound Poisson process. Loubergé, Kellezi and Gilli (1999) pricing a loss index triggered cat bond applying the catastrophe option pricing model developed by Cummins and Geman (1995). Lee and Yu (2002) develop a contingent claim model to price a CAT bond through geometric Brownian motion. This model incorporates stochastic interest rates and considers moral hazard, basis risk and default risk. Biagini, Bregman and Meyer-Brandis (2008) value catastrophe options and describe the index using an inhomogeneous compound Poisson process for the loss period and use an inhomogeneous exponential Levy process to re-estimate the index during the development period and up to maturity.

Jaimungal and Wang (2006) analyze the pricing of catastrophic put options under stochastic interest rates with losses generated by a compound Poisson process. Asset prices are modeled through a jump-diffusion process which is correlated to the loss process. To evaluate these catastrophe options Wang (2016) employ a compound doubly stochastic Poisson process with lognormal intensity to describe accumulated losses and assume the volatility varies stochastically. Jarrow (2010) use a pricing methodology based on the reduced form models used to price credit derivatives. Nowak and Romaniuk (2013) apply TSIR models if the occurrence of the catastrophe does not depend on the financial market's behavior. Zong-Gang and Chao-Qun (2013) derive a bond pricing formula in a stochastic interest rates environment with the losses following a compound nonhomogeneous Poisson process. Braun (2011) proposes a catastrophic swap pricing model representing the occurrence of catastrophes through a doubly stochastic Poisson process (Cox process) with a mean-reverting Ornstein-Uhlenbeck intensity. Lai, Parcollet and Lamond (2014) calculate the price of a catastrophe bond from a jump-diffusion process representing catastrophes, a three-dimensional stochastic process to represent the exchange rate, domestic and foreign interest rates, and the hedging cost for the currency risk.

Pérez-Fructuoso (2008) and Pérez-Fructuoso (2009) developed a new model for calculating the catastrophe loss index, whose value is the sum of the reported loss amount for each event. This variable is calculated as the difference between the total catastrophe's severity and the key variable in the model named the incurred but not yet reported claims amount which is driven by a geometric Brownian motion with a constant claims reporting rate.

To develop a more precise expression of the catastrophic loss ratio, Pérez-Fructuoso (2016) develops an alternative model whose central hypothesis is that the incurred but not yet reported claims amount decreases proportionally to an exponential function, called the asymptotic claims reporting rate. The dynamics of this decrease is represented by a geometric Brownian motion.

Finally, and with the same objective as in the previous case, Pérez-Fructuoso (2017), models the decreasing linear dynamics of incurred but not yet reported claims amount, by means of an additive Brownian process or Ornstein-Uhlenbeck process.

Finally, Pérez-Fructuoso (2022) makes a comparison of the three models mentioned above. With the available data, it can be concluded that Ornstein-Uhlenbeck model is the one that fits better the real-life claims reporting process. However, we have also seen that the asymptotic model fits well the first two weeks after the catastrophes occurred.

3. METHODOLOGY

3.1. Catastrophe loss index definition

A catastrophe loss index can be defined as the quotient between the total losses from catastrophes occurred over the period $[0,T]$ (risk period) and a constant value p, whose definition depends upon the kind of index to be employed. The index value at maturity T', $LI(T')$, with $T' \geq T$, is defined as:

$$LI(T') = \frac{L(T')}{p} \tag{1}$$

where $L(T')$ is the total claims reported in T' for all catastrophes occurring during the risk period.

$L(T)$ is a random variable that depends on the following risk factors:

- The number of catastrophes, $N(T)$, occurring during the risk period, $[0,T]$.
- The moment of time τ_j when the catastrophe j occurs, for $j = 1, \ldots, N(T)$ and $\tau_j \in [0,T]$.
- The severity of each catastrophic event K_j, for $j = 1, \ldots, N(T)$.
- The reported loss process behavior, $S_j(t)$, representing the reported loss amount at time t associated with the catastrophe j, for $j = 1, \ldots, N(T)$ and $t \in [\tau_j, T']$.

Under these assumptions, the numerator of the catastrophe loss index can be calculated as:

$$L(T') = \sum_{j=1}^{N(T)} S_j(T') \tag{2}$$

3.2. Modelling hypotheses

To obtain the value of (1), we can assume some hypotheses for our modelling.

It is well known in probability theory that; a Poisson process is a stochastic process that counts the number of events that happen in a certain period $[0,T]$. Therefore, we assume that $N(T)$ is a Poisson process,

$$N(T) \sim Poisson(\lambda)$$

where λ is the average number of catastrophic events per period $[0,T]$. The time between two different events in a Poisson process has an exponential distribution with the same parameter. Then, for our modelling:

$$t_{j+1} - t_j \sim Exp(\lambda)$$

If K_j is the total amount of the catastrophe j and $S_j(t)$ is for the reported loss amount at time t, we define $R_j(t)$ as the incurred-but-not-yet-reported (IBNRL) loss amount at time t associated to the catastrophe j. Thus,

$$S_j(t) = K_j - R_j(t)$$

and the numerator of the catastrophe loss index (1) becomes:

$$L(T') = \sum_{j=1}^{N(T)} \left(K_j - R_j(T')\right) \tag{3}$$

In the following section, we develop the model that allows us to obtain an expression of $R_j(t)$ to calculate the reported loss amount, and therefore the numerator of the catastrophe loss index triggering Cat Bonds. We consider the occurrence of a single generic catastrophe, K, occurring at moment $\tau \in [0,T]$, and we assume that its claims claim reporting process is $S(t)$, and $R(t)$ is its occurred but not yet reported loss process.

3.3. Formulation model

We define catastrophe severity as the sum of two random variables,

$$K = S(t) + R(t) \tag{4}$$

were $R(t)$ represents the incurred-but-not-yet-reported loss (IBNRL) amount at time t associated to the catastrophe occurred in τ and $S(t)$ stands for the reported loss amount at time t related to a single catastrophe occurred in τ.

Once a catastrophe of severity K has occurred in τ, the claim reporting process associated with this single catastrophe is initiated lasting until the end of the maturity period. Empirical evidence shows that the intensity of claims reporting would seem to be greatest just after the occurrence of the catastrophe and decreases with time.

Then, the instantaneous claim reporting process is represented by a differential equation that describes the increase of reported loss amount proportional to the incurred-but-not-yet-reported loss amount, the main variable of the formal model,

$$dS(t) = \alpha(t-\tau) \cdot R(t)dt \tag{5}$$

were $\alpha(t-\tau)$ is a real-value function named *claim reporting rate function*.

By differentiating equation (5) results,

$$dS(t) = -dR(t) \tag{6}$$

and by substituting (6) into equation (5), the differential equation that describes the evolution of $R(t)$ is obtained:

$$dR(t) = -\alpha(t-\tau) \cdot R(t)dt \tag{7}$$

Equation (7) shows that the incurred-but-not-yet-reported loss amount in t decreases with time according to the claim reporting rate function.

In order to capture the irregular behavior of the claim reporting process, we introduce a Wiener process (Brownian motion) into equation (7). This irregularity in the claim reporting process depends on the IBNRL amount. While there is still much to declare, the irregularity of the declarations is high. However, it decreases as the IBNRL amount does it too. To reflect this behavior, we will add a Wiener process with intensity $\sigma R(t)$, which is called a geometric Brownian motion. The result of adding it into our model (7) is the following stochastic differential equation,

$$dR(t) = -\alpha(t-\tau) \cdot R(t)dt + \sigma \cdot R(t) \cdot dW(t) \tag{8}$$

where $\alpha(t-\tau)$ is the claim reporting rate function that represents the process' drift, σ is a constant value that represents the process' volatility and W_t is a standard Wiener process associated to the catastrophe.

A necessary condition for our modelling lies in the fact that σ must be a low value. Otherwise, if σ is large enough, the claim reporting rate might become positive and then the IBNRL amount will also grow, unlike our initial assumption.

Applying Itô's Lemma (Friedman (1975); Malliaris and Brock (1991); Arnold (1974)) in equation (8), we obtain the expression for the incurred-but-not-yet-reported loss amount, $R(t)$:

$$R(t) = K \cdot \exp\left(-\int_0^{t-\tau} \alpha(s)ds - \frac{\sigma^2}{2} \cdot (t-\tau) + \sigma \cdot W(t-\tau)\right) \quad (9)$$

with the following boundary conditions:

- Initial boundary conditions: if $t = \tau$ then $R(t) = K$, the incurred-but-not-yet-reported loss amount is the catastrophe severity.
- Final boundary condition: if $t \to \infty$ then $R(t) = 0$, the catastrophe incurred loss is reported and, obviously, the incurred-but-not-yet-reported loss amount is zero.

From the relation defined between $R(t)$ and $S(t)$ defined in equation (4), we obtain $S(t)$ as follows:

$$S(t) = K \cdot \left[1 - \exp\left(-\int_0^{t-\tau} \alpha(s)ds - \frac{\sigma^2}{2} \cdot (t-\tau) + \sigma \cdot W(t-\tau)\right)\right] \quad (10)$$

Symmetrically to the incurred-but-not-yet-reported loss amount, the boundary conditions for the reported loss amount are:

- Initial boundary conditions: if $t = \tau$ then $S(t) = 0$, the reported loss amount is zero.
- Final boundary condition: if $t \to \infty$ then $S(t) = K$, the reported loss amount es the catastrophe severity.

Notice that is straightforward to see that if $\sigma = 0$ we draw as a result the expression for both the incurred-but-not-yet-reported loss amount and the reported loss amount in a deterministic model:

$$R(t) = K \cdot \exp\left(-\int_0^{t-\tau} \alpha(s)ds\right) \text{ and } S(t) = K \cdot \left[1 - \exp\left(-\int_0^{t-\tau} \alpha(s)ds\right)\right] \quad (11)$$

When the claim reporting rate is defined in a hybrid form, we assume that this rate is increasing up to a certain point in time, which we symbolize as t_m, and thereafter this rate remains constant at the level α until the claims reporting process ends:

$$\alpha(s) = \begin{cases} \frac{\alpha_m}{t_m} \cdot s & 0 \leq s \leq t_m \\ \alpha_m & s > t_m \end{cases} \quad (12)$$

3.4. Model parameters estimation through evolutionary strategies

In proposed model to represent the claim process, the claim reporting rate is the variable that should be calculated for each set of real data. The model of claim reporting rate is defined through three parameters (α_m, σ, t_m). Thus, the global optimization procedure must simultaneously adjust all of them. The main goal of the global optimization problem is summarized in the following definition (Törn, 1991):

Given a function: $f: M \subseteq \Re^n \to \Re, M \neq \emptyset,$

for $x \in M$ the value $f^* := f(x^*) > -\infty$ is a global minimun, iff: $\forall x \in M: f(x^*) \leq f(x)$

Then x^* is a global minimum point, f is called objective function, and the set M is called the feasible region. In this case, the global optimization problem has a unique restriction: the claim reporting rate must be positive. This restriction is included in the codification and all individuals are processed to become feasible ones. Then, despite this restriction, the solutions space does not have infeasible regions.

Evolutionary Algorithms are the term used to describe a broad set of algorithms that draw inspiration from the biological process of natural selection. Examples of evolutionary algorithms include genetic algorithms, genetic programming, evolutionary strategies, and differential evolution. A notable feature that explains their success when applied to optimization problems is their ability to strike an appropriate balance between exploration and exploitation during the search process. Evolutionary Algorithms combine characteristics of both classifications of classical optimization techniques, volume-oriented and path-oriented methods. Volume-oriented methods (Monte-Carlo strategies, clusters algorithms) apply the searching process scanning the feasible region while path-oriented methods (pattern search, gradient descent algorithms) follow a path in the feasible region. A definition of a restricted search space of finite volume and the starting point is required to volume-oriented and path-oriented methods respectively.

Evolution strategies (ES) developed by Rechenberg (1971) and Schwefel (1981), have been traditionally used for optimization problems with real-valued vector representations. As Genetic Algorithms (GA) this are heuristics search techniques based on the building block hypothesis. Unlike GA, however, the search is basically focused on the gene mutation. This is an adapting mutation based on the likely the individual represents the problem solution. The recombination also plays an important role in the search, mainly in adapting mutation. ES are techniques widely used (and more appropriated than GA) in real-values optimization problems. ES offer practical advantages facing difficult optimization problems (Fogel, 1997). These advantages are: its conceptual simplicity, broad applicability, potentiality to use knowledge and hybridize with other methods, implicit parallelism, robustness to dynamic changes, capability for self-optimization and capability to solve problems that have no known solutions. A general ES is defined as an 8-tuple (Bäck, 1996):

$$ES = (I, \Phi, \Omega, \Psi, s, \iota, \mu, \lambda)$$

where:

- $I = (\vec{x}, \vec{\sigma}, \vec{\alpha}) = \Re^n \times \Re_+^{n_\sigma} \times [-\pi, \pi]^{n_\alpha}$ is the space of individuals, i.e. the parameters of the model, α_m, σ and t_m,
- $n_\sigma \in \{1, \dots, n\}$ represents the dimension of the vector of standard deviations of the parameters to be adjusted and n is the number of parameters to be fitted,
- $n_\alpha \in \left\{0, \frac{(2n-n_\sigma)(n_\sigma-1)}{2}\right\}$ is the dimension of the vector of rotation angles,
- $\Phi: I \to \Re = f$ is the fitness function,
- $\Omega = \{m_{\{\tau,\tau\prime,\beta\}}: I^\lambda \to I^\lambda\} \cup \{r_{\{rx,r\sigma,r\alpha\}}: I^\mu \to I^\lambda\}$ are the genetic operator, mutation o recombination operators,
- $\Psi(P) = s\left(P \cup m_{\{\tau,\tau\prime,\beta\}}\left(r_{\{rx,r\sigma,r\alpha\}}(P)\right)\right)$ is the process to generate a new set of individuals,
- s is the selection operator and
- ι is the termination criterion.

In this work, the definition of the individual has been simplified: the rotation angles n_α have not been considered, $n_\alpha = 0$.

The mutation operator generates new individuals as follows:

$$\sigma_i' = \sigma_i \cdot exp(\tau' \cdot N(0,1) + \tau \cdot N(0,1))$$

$$\vec{x}' = \vec{x} + \sigma_i' \cdot \vec{N}(\vec{0}, 1)$$

ES has several formulations, but the most common form is (μ, λ)-ES, where $\lambda > \mu \geq 1$, (μ, λ) means that μ-parents generate λ-offspring through recombination and mutation in each generation. The best μ offspring are selected deterministically from the λ offspring and replace the current parents. Elitism and stochastic selection are not used. ES considers that strategy parameters, which roughly define the size of mutations, are controlled by a "self-adaptive" property of their own. An extension of the selection scheme is the use of elitism; this formulation is called $(\mu + \lambda)$-ES. In each generation, the best μ-offspring of the set μ-parents and λ-offspring replace current parents. Thus, the best solutions are maintained through generation. The computational cost of (μ, λ)-ES and $(\mu + \lambda)$-ES formulation is the same.

4. ANALYSIS OF RESULTS

4.1. Solution of the proposed model for a hybrid claim reporting rate

In the case where the claim reporting rate is defined in a hybrid form, we calculate the integral of equation (9) in two different situations:

1) If the valuation moment, t, is previous than the moment of claim reporting rate change, $\tau \leq t \leq \tau + t_m$, then:

$$\int_0^{t-\tau} \alpha(s)ds = \int_0^{t-\tau} \left(\frac{\alpha_m}{t_m} \cdot s\right) ds = \frac{\alpha_m}{t_m} \cdot \frac{s^2}{2}\bigg]_0^{t-\tau} = \frac{\alpha_m \cdot (t-\tau)^2}{2t_m} \quad (13)$$

   By substituting this result into equation (9), the incurred-but-not-yet-reported loss amount at t results,

$$R(t) = K \cdot \exp\left[-\left(\frac{\alpha_m}{2t_m} \cdot (t-\tau) + \frac{\sigma^2}{2}\right) \cdot (t-\tau) + \sigma \cdot W(t-\tau)\right] \quad (14)$$

   and the reported loss amount until t is:

$$S(t) = K \cdot \left(1 - \exp\left[-\left(\frac{\alpha_m}{2t_m} \cdot (t-\tau) + \frac{\sigma^2}{2}\right) \cdot (t-\tau) + \sigma \cdot W(t-\tau)\right]\right) \quad (15)$$

2) If the valuation moment, t, is after than the moment of claim reporting rate change, $t > \tau + t_m$, then:

$$\int_0^{t-\tau} \alpha(s)ds = \int_0^{t_m} \left(\frac{\alpha_m}{t_m} \cdot s\right) ds + \int_{t_m}^{t-\tau} \alpha_m ds = \frac{\alpha_m}{t_m} \cdot \frac{s^2}{2}\bigg]_0^{t_m} + \alpha_m \cdot s\big]_{t_m}^{t-\tau} = \alpha_m \cdot (t-\tau) - \frac{\alpha_m \cdot t_m}{2} \quad (16)$$

   By substituting this result into equation (9), the incurred-but-not-yet-reported loss amount at t results,

$$R(t) = K \cdot \exp\left[-\left(\alpha_m + \frac{\sigma^2}{2}\right) \cdot (t-\tau) + \sigma \cdot W(t-\tau)\right] \cdot \exp\left(\frac{\alpha_m \cdot t_m}{2}\right) \quad (17)$$

   and the reported loss amount until t is:

$$S(t) = K \cdot \left(1 - \exp\left[-\left(\alpha_m + \frac{\sigma^2}{2}\right) \cdot (t-\tau) + \sigma \cdot W(t-\tau)\right] \cdot \exp\left(\frac{\alpha_m \cdot t_m}{2}\right)\right) \quad (18)$$

$R(t)$ follows a log-normal distribution, where normal distribution parameters associated are:

- If $\tau \leq t \leq \tau + t_m$, then:

$$\ln R(t) \sim N\left(\ln K - \left(\frac{\alpha_m \cdot (t-\tau)}{2t_m} + \frac{\sigma^2}{2}\right) \cdot (t-\tau); \sigma^2 \cdot (t-\tau)\right)$$

- If $t > \tau + t_m$, then:

$$\ln R(t) \sim N\left(\ln K - \left(\alpha_m + \frac{\sigma^2}{2}\right) \cdot (t-\tau) + \frac{\alpha_m \cdot t_m}{2}; \sigma^2 \cdot (t-\tau)\right)$$

4.2. Catastrophe loss index calculation

As noted in section 3.1, the catastrophe loss ratio is defined by equation (1). For the case of catastrophe bonds, the indices used as triggers for indemnity payments are based on the accumulated losses to maturity associated with a single catastrophe. Therefore, we define the Bernoulli variable or indicator variable, δ, which is 0 if the catastrophe covered in the issue does not occur or 1 otherwise and the loss ratio is rewritten as:

$$LI(T') = \delta \frac{S(T')}{p} = \begin{cases} 0 & \text{si } \delta = 0 \\ \frac{S(T')}{p} = \frac{K - R(T')}{p} & \text{si } \delta = 1 \end{cases} \quad (19)$$

When the claim reporting rate is defined through a hybrid model, the catastrophe loss index at maturity is given by the following expression:

$$LI(T') = LI(t_m) + LI(T' - t_m) =$$

$$= \delta \cdot \frac{K}{p} \cdot \left[\begin{array}{c} \left(1 - \exp\left(-\left(\frac{\alpha_m \cdot (t_m - \tau)}{2t_m} + \frac{\sigma^2}{2}\right) \cdot (t_m - \tau) + \sigma \cdot W(t_m - \tau)\right)\right) + \\ + \left(1 - \exp\left(e^{-\left(\alpha_m + \frac{\sigma^2}{2}\right) \cdot (T' - t_m) + \sigma \cdot W(T' - t_m)} \cdot e^{\frac{\alpha_m \cdot t_m}{2}}\right)\right) \end{array} \right] \quad (20)$$

4.3. Adjusting claims reporting rate, volatility and moment t_m

In this work, we have a mathematical model that must be fitted to real data samples obtained from real catastrophes. Then, the problem could be defined as: "search for parameters that define a function that minimizes the error for each real data sample".

The proposed model follows equation (15) and (18), in both cases three parameters (α_m, σ and t_m) should be calculated to obtain the minimum distance to the real data distribution. The fitness function (evaluation of everyone over each series) is clearly the sum of the squared errors over the real data set but, in this case, due to the stochastic nature of the function, each individual must be evaluated by calculating the average value of the error over a large number of experiments repetitions. In this work, 10,000 evaluations have been performed to eliminate the randomness introduced by the Wiener process.

The type of recombination used in this work is the discrete recombination and the strategy $(\mu + \lambda)$-ES was used to select the individual to the next generation.

The algorithms used in this work have been developed by the authors in the R language and with the support of the cmaesr package (Bossek, 2021). The execution has been carried out on several computers Intel Core i5 4.8GHz with Windows 11 operating system. We have three computers with these characteristics to carry out the work. We performed 10,000 evaluations in each city to have the data available in one week. This number of evaluations is more than enough to ensure that the average value obtained is significant.

So, considering that we have data associated with seven cities and that each city has an execution time of 10000 evaluations of 1 week, the total execution time has been 3 weeks (first week three cities, second week three cities and third week the remaining city).

The parameters of the ES are summarized in Table 1. Besides, different runs were achieved changing the random seed[1].

Table 1: Setting of exogenous parameters of the ES. Source: Own elaboration.

Parameter	Value
Initial standard deviations	Randomly generated in range [0.0;3.0]
Number of rotation angles	0
Parent population	20
Offspring population size	80
Termination criterion	Number of generation step= 500

The real data to adjust model is related with floods occurred in several Spanish regions: Alcira (10/01/1991), San Sebastián (06/23/1992), Barcelona 1 (09/14/1999), Barcelona 2 (10/20/2000), Murcia (10/20/2000), Valencia (10/20/2000) and Zaragoza (10/20/2000). The data are temporal series of week incurred-but-not-yet-reported loss. For each disaster, we apply the optimization procedure for the model.

[1] Seeds are generated sequentially for a linear congruent random number generator. Careful choice of sedes is not necessary.

Week	**Alcira** (10/01/1991)	**San Sebastián** (06/23/1992)	**Barcelona 1** (09/14/1999)	**Barcelona 2** (10/20/2000)	**Murcia** (10/20/2000)	**Valencia** (10/20/2000)	**Zaragoza** (10/20/2000)
0	100	100	100	100	100	100	100
1	84.94	88.08	90.68	85.95	88.46	97.54	60.11
2	53.65	36.04	68.38	61.73	75.55	80.18	56.91
3	34.96	23.67	50.68	34.98	48.7	60.15	38.3
4	24.05	16.68	41.42	24.07	31.13	43.16	29.79
5	18.86	12.29	31.58	14.05	21.41	31.96	23.4
6	13.36	9.94	25.43	12.41	15.78	27.55	19.15
7	10.53	8.72	19.56	8.52	11.27	19.64	18.09
8	8.04	7.76	16.68	5.23	8.71	15.29	15.43
9	6.94	6.8	13.28	5.23	8.24	14.76	15.43
10	5.23	5.78	10.54	5.08	6.57	14.7	10.64
11	4.08	5.18	8.15	3.44	5.36	11.06	6.91
12	3.71	4.33	6.8	3.59	4	8.46	3.72
13	3.56	3.45	6.13	2.24	3.38	6.98	3.72
14	2.6	2.69	3.41	1.2	2.6	6.21	2.66
15	1.75	1.81	3.41	0.6	2.8	5.17	2.66
16	1.3	1.59	2.61	0.6	2.29	4.22	1.6
17	0.77	1.39	1.81	0.6	2.25	3.5	1.6
18	0.29	1.16	1.26	0.45	2.14	2.72	1.6
19	0	0.96	0.56	0.45	1.67	2.26	0
20		0.76	0	0.45	1.28	1.88	
21		0.45		0.45	1.09	1.69	
22		0.28		0	0.93	1.61	
23		0.2			0.66	0.9	
24		0.17			0.66	0.54	
25		0.11			0.62	0.36	
26		0.06			0.16	0.19	
27		0			0	0	

Table 2: Incurred-but-not-yet-reported loss amount weekly in (%): Own elaboration

These data have been elaborated by the Technical and Reinsurance Department of the Consorcio de Compensación de Seguros (a public institution dependent on the Spanish Ministry of Economic Affairs and Digital Transition) to be applied exclusively in this research. The way in which they are presented, as a percentage of the weekly reported loss amount, means that they are not affected by the passage of time. That is, the data expressed as a percentage allows avoiding the time gap for its use at different moments in time. The amount of the catastrophe could be higher today, but the percentage declared in the first week would remain approximately the same (mainly because the population, public infrastructure and housing have not been modified in the affected areas after reconstruction).

Table 3 shows global results over the set of catastrophes for proposed model with evolutionary strategies, the accumulative quadratic error for each catastrophe and for the model. For each catastrophe, the model has different parameters (the best parameters for this real data distribution). Finally, the accumulative, the average and the standard deviation of the error for all catastrophes are shown.

Table 3: Global results. Source: Own elaboration.

Alcira (10/01/1991)	1059.89
San Sebastián (06/23/1992)	1060.43
Barcelona 1 (09/14/1999)	537.39
Barcelona 2 (10/20/2000)	472.15
Murcia (10/20/2000)	624.73
Valencia (10/20/2000)	1147.68
Zaragoza (10/20/2000)	687.35
Accumulative Quadratic Error	5589.61
Average Error	798.52
Standard Deviation	281.69

For each event, the optimization procedure described above is applied. Table 4 shows the parameters of the proposed model associated with each of the catastrophes considered:

Table 4: Parameters. Source: Own elaboration.

	α_m	σ	t_m
Alcira (10/01/1991)	0.440	0.096	0.002
San Sebastián (06/23/1992)	0.241	0.053	0.733
Barcelona 1 (09/14/1999)	0.248	0.039	0.042
Barcelona 2 (10/20/2000)	0.317	0.042	0.190
Murcia (10/20/2000)	0.058	0.018	0.684
Valencia (10/20/2000)	0.027	0.029	0.658
Zaragoza (10/20/2000)	0.168	0.011	0.530

5. DISCUSSION

In most of the previous models analyzed, a geometric Wiener process is assumed to model the reported loss amount. This assumption implies an exponential growth, on average, of the instantaneous claim reporting rate within the interval considered. In many models (e.g. Cummins and Geman (1995); Lee and Yu (2002); Wang (2016); Zong-Gang and Chao-Qun (2013); Lai, Parcollet and Lamond (2014)), this rate is further assumed to be discontinuous by introducing the jump process due to major catastrophes in the definition of $S(t)$; in others models (e.g., Geman and Yor (1997); Aase (1999); Embrechts and Meister (1997); Muermann (2003); Biagini, Bregman and Meyer-Brandis (2008); Jaimungal and Wang (2006); Braun (2011)), the introduction of major catastrophes is done directly in the definition of the loss ratio. This aggregate approach to the behavior of the claims' reporting intensity does not correspond to a uniform distribution of claims occurrence within a specific interval, as it is difficult to understand that the aggregation process is exponential and not linear.

A geometric Brownian motion is also considered to model the behavior of the Cat Bond loss index trigger. However, unlike previous models, it uses the Wiener process to explain the decreasing dynamics of the incurred-but-not-yet-reported loss amount, rather than to describe the evolution of the reported loss amount, which is obtained by subtraction of the former from the total severity of the specified catastrophe. The loss index is then the reported loss amount multiplied by an indicator which varies according to the likelihood of the catastrophic event occurring, thus notably simplifying both the calculation of the index and the estimation of the parameters.

On the other hand, it is important to note that the model proposed here is limited to determining the numerator of the catastrophe loss rate without going into the valuation or pricing expressions for catastrophe bonds. However, the price of the catastrophe bond at time t of its trading period can be calculated by applying general option pricing theory (see, for example, Loubergé, Kellezi and Gilli (1999); Baryshnikov, Mayo and Taylor (2001); Burneki and Kukla (2003) or Nowak and Romaniuk (2013)). Likewise, the methodology used by Jarrow (2010) could also be used to value the bond, so that in this case it would only be necessary to determine the probability of occurrence of the catastrophe and the time structure of the LIBOR interest rates, values that are normally calculated by specialized modeling agencies.

This article is an extension of previous works developed by Pérez-Fructuoso (2008), Pérez-Fructuoso (2016) and Pérez-Fructuoso (2017). All of them are based on a growth of the reported loss amount proportional to the incurred-but-not-yet-reported loss amount, which is the fundamental modeling variable. The proportionality function, called the claim reporting rate, is the one that varies according to the proposed model. In Pérez-Fructuoso (2008), the claim reporting rate is assumed to be constant. In Pérez-Fructuoso (2016) it is defined asymptotically, i.e., it is assumed to tend to a constant value. Specifically, at the beginning of the process it takes the value zero to, subsequently, grow until it reaches the constant value. Finally, in Pérez-Fructuoso (2017) it is assumed that the intensity of the irregularity in the reported loss amount is constant over time and does not depend on the incurred-but-not-yet-reported loss amount. To reflect this in the model, an arithmetic Brownian motion is used instead of a geometric one.

After calculating the predictions associated with the available data, it was concluded that the model proposed by Pérez-Fructuoso (2017) best fit the real claim reporting process. However, it was also observed that the model developed by Pérez-Fructuoso (2016) fitted the data well during the first two weeks after the flood occurred. Therefore, it was proposed as future work to create a new model that would consider a mixed claim reporting rate, as proposed here, increasing during the first weeks and constant from a certain point until the end of the reporting process.

Concerning the estimation of the model parameters, the technique used here, a machine learning technique from the area of Artificial Intelligence, has not been applied in any of the previous works related to the analyzed topic.

Within the field of machine learning there is a set of methods based on natural processes such as natural selection, social behavior, genetics, neural processes, etc. In particular, the method used in this article falls within the so-called Evolutionary Computation (Holland, 1975).

Evolutionary Computation techniques use the process of evolution of species proposed by Darwin as the basis for the development of search and optimization algorithms. There are different algorithms depending on the problem to be solved. In this case the values to be adjusted are real numbers, so Evolutionary Strategies (Schwefel, 1988) that work directly with real numbers are applied. This type of strategy allows the search to be performed without incorporating knowledge of the problem, i.e., the search algorithm does not need to know how the models are defined, it only needs to know the results of the models (the error function to be minimized). In this way, the strategy evaluates the possible solutions (specific values for the model parameters), selects those that are better (have obtained a lower error value on the catastrophe data series) and from this selection generates new possible solutions (applying operators typical of the evolutionary strategy) until the solution that minimizes the objective function is found.

6. CONCLUSION

The proposed model for the distribution of the total loss amount allows for a simple calculation of the loss index trigger for catastrophe bonds. The core of the model is the definition of the decreasing dynamics of the variable incurred-but-not-yet-reported loss amount based on a mixed model in which the claim reporting rate is defined as increasing up to a certain time and constant thereafter until the end of the reporting period considered. The claim reporting rate is random and is modeled by a geometric Wiener process to adequately represent the real reported loss amount. The reported loss amount, numerator of the loss ratio, is easily obtained by the difference between the catastrophe total loss amount and the incurred-but-not-yet-reported loss amount.

The relative simplicity of the presented model eases parameter estimation and simulation. In this work, the application of a machine learning techniques allows to estimate the parameters of the model by the optimization of the accumulative quadratic error. These techniques facilitate the estimation process for this type of applications where appropriate global parameters are not available to explain the whole data set, but specific parameters are needed to describe subsets corresponding to the same model in different situations.

It should be noted that the available data are very specific to a geographical location such as Spain, whose meteorological characteristics are very different from those of the USA (a flood in Spain is far from being a hurricane in the USA), so that it is not possible a priori to extrapolate the results obtained on the adequacy of the models. This is because insurance companies do not disclose any information on all the available data they have, so it is difficult to obtain more data, which would be useful to test the validity of the proposed model and would allow more generalized real conclusions to be drawn.

To continue this project, the first step to be taken is to obtain more real data. Then, it will be possible to test more accurately the proposed model and the previous models. However, if we want to go further, the models proposed by Pérez-Fructuoso (2008), (2016), (2017) and this one with stochastic volatility should be studied. Stochastic volatility seems to fit very well in many financial models, although it is complex to implement.

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